Note: for conservative body forces (e.g., gravity) $\rho b_i = \frac{\partial \psi}{\partial x_i}$ and the force term can be absorbed into the pressure term by modifying it to $\tilde{P} = P - \psi$.

Definition 4.5. *For fluids, a flow is* ***steady*** *if* $\frac{\partial}{\partial t}\mathbf{v}(\mathbf{x}, t) = \mathbf{0}$*. In this case, the equations of linear momentum balance* (4.15) *are*

$$\rho \left(v_k \frac{\partial}{\partial x_k} \right) v_i = -\rho b_i + \frac{\partial T_{ij}}{\partial x_j}.$$

Definition 4.6. *A continuum is in* ***equilibrium*** *if* $\mathbf{v} = 0$.

In this case, the equations of linear momentum balance (4.15) *yield*

$$\rho b_i + \frac{\partial}{\partial x_j} T_{ij} = 0.$$

Remark 4.7. *The force exerted by a continuum on a body B contained within it (e.g, a body immersed in a fluid) can be determined by integrating the Cauchy stress vector over the boundary of the body. Hence, this force is given by*

$$\int_{\partial B} \mathbf{t}\, dA = \int_{\partial B} \mathbf{T}\mathbf{n}\, dA.$$

5 Properties of solutions of the Euler Equations

The main example of a continuum theory which we study in this course will be the Euler equations for flow of an incompressible, ideal fluid:

$$\rho_0 \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla P + \rho_0 \mathbf{b}\ . \tag{5.1}$$

We assume that the body force is conservative so that $\mathbf{b} = -\nabla \chi$ (for example, gravity corresponds to $\chi = g\,\mathbf{x}.\mathbf{e}_3$, where g is the acceleration due to gravity). The conservation of mass equation (3.4) then reduces to the incompressibility condition

$$\nabla \cdot \mathbf{v} = 0 \ \text{ in } \Omega. \tag{5.2}$$

Recall the important concept in the study of fluid flows of the *vorticity* $\boldsymbol{\omega}(\mathbf{x}, t) = (\omega_i(\mathbf{x}, t))$, defined by

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}. \tag{5.3}$$

This is a measure of the rotation inherent in the flow. We say that the flow is *irrotational* if the vorticity (5.3) is identically zero.

5.1 Bernoulli's Theorem

Theorem 5.1. *Let $\mathbf{v} = \mathbf{v}(\mathbf{x})$ be a steady solution of the incompressible Euler equations and let the body force satisfy $\boldsymbol{b} = -\nabla \chi$. Then*

$$H = \left(\frac{P}{\rho_0} + \chi + \frac{1}{2}|\mathbf{v}|^2 \right)$$

is constant along streamlines of the flow.

Proof. See Problem sheet 5, Q3. □

Example 5.2 (Applications of Bernoulli's Theorem).

5.2 Kelvin's Circulation Theorem

This states that the circulation around a closed curve in an inviscid incompressible fluid flowing according to the Euler equations (5.1) is constant.

Theorem 5.3. *Let $\boldsymbol{\phi}(\mathbf{x}, t)$ be a motion, $\boldsymbol{\phi} : \Omega \times [0, T] \to \mathbb{R}^3$ and suppose that the corresponding spatial velocity field $\mathbf{v}(\mathbf{x}, t)$ satisfies the Euler equations* (5.1)*. Let C be a closed simple curve in the initial configuration Ω and let C_t denote its image at time t under the motion (so that $C_t = \boldsymbol{\phi}(C, t)$). Then the circulation around C_t defined by*

$$\Sigma(t) = \int_{C_t} \mathbf{v}.d\mathbf{x}$$

satisfies

$$\frac{d\Sigma(t)}{dt} = 0.$$

Proof. This follows by Corollary 3.6 and (5.1): □

5.3 Helmholtz Theorems on Vorticity

Definition 5.4. *A vortex line at time t is a curve $\mathbf{y}(s)$, $\mathbf{y} : [a, b] \to \Omega_t$ such that $\mathbf{y}(s)$ is tangent to the vorticity vector $\boldsymbol{\omega}(\mathbf{x}, t)$ $\mathbf{x} \in \Omega_t$ and so satisfies*

$$\frac{d\mathbf{y}(s)}{ds} = \boldsymbol{\omega}(\mathbf{y}(s), t), \quad s \in [a, b]. \tag{5.4}$$

A vortex tube is a surface made up of the vortex lines passing through a closed contour.

Remark 5.5. ***(Helmoltz result on conservation of vorticity flux.)*** *Consider a contour curve C, oriented as shown and with corresponding spanning surface S.*

Then, by Stokes Theorem,

$$\oint_C \mathbf{v}.d\mathbf{r} = \int_S (\nabla \times \mathbf{v}).\mathbf{n}\, dA = \int_S \boldsymbol{\omega}.\mathbf{n}\, dA, \quad . \tag{5.5}$$

We call $\int_S \boldsymbol{\omega}.\mathbf{n}\, dA$ the flux of vorticity *through the surface S and it equals the circulation around the corresponding contour C.*

Now consider a vortex tube[6] *as illustrated with end curves C_1, C_2 and bounding surfaces S_1, S_2 as illustrated. Let $\tilde{\Omega}$ be the 3D-domain bounded by S_1, S_2 and the vortex tube surface S_3. Because we have a vortex tube, it follows that $\boldsymbol{\omega}.\tilde{\mathbf{n}} = 0$ on S_3 where $\tilde{\mathbf{n}}$ is the outward unit normal from $\tilde{\Omega}$. Hence by the Divergence Theorem*

$$0 = \int_{\tilde{\Omega}} \nabla.\boldsymbol{\omega}\, dV = \int_{\partial\tilde{\Omega}} \boldsymbol{\omega}.\tilde{\mathbf{n}}\, dA = -\int_{S_1} \boldsymbol{\omega}.(-\tilde{\mathbf{n}}) + \int_{S_2} \boldsymbol{\omega}.\tilde{\mathbf{n}} + \int_{S_3} \boldsymbol{\omega}.\tilde{\mathbf{n}}$$

where the third integral over S_3 is zero, and hence, using (5.5),

$$\oint_{C_1} \mathbf{v}.d\mathbf{r} = \int_{S_1} \boldsymbol{\omega}.(-\tilde{\mathbf{n}})\, dA = \int_{S_2} \boldsymbol{\omega}.\tilde{\mathbf{n}}\, dA = \oint_{C_2} \mathbf{v}.d\mathbf{r}.$$

This illustrates that if the tube narrows, then the vorticity flux through a cross-section of the tube at that point must increase.

Remark 5.6. *We will show, using the next two results, that vortex lines are transported by the flow of an incompressible, ideal fluid.*

[6] i.e., a tube in the flow whose sides are made up of vortex lines.